

TOPOLOGICAL ASPECTS OF HOLOMORPHIC MAPPINGS OF HYPERQUADRICS FROM \mathbb{C}^2 TO \mathbb{C}^3

MICHAEL REITER

ABSTRACT. Based on the results in [Rei14a] we deduce some topological results concerning holomorphic mappings of Levi-nondegenerate hyperquadrics under biholomorphic equivalence. We study the class \mathcal{F} of so-called nondegenerate and transversal holomorphic mappings sending locally the sphere in \mathbb{C}^2 to a Levi-nondegenerate hyperquadric in \mathbb{C}^3 , which contains the most interesting mappings. We show that from a topological point of view there is a major difference when the target is the sphere or the hyperquadric with signature $(2, 1)$. In the first case \mathcal{F} modulo the group of automorphisms is discrete in contrast to the second case where this property fails to hold. Furthermore we study some basic properties such as freeness and properness of the action of automorphisms fixing a given point on \mathcal{F} to obtain a structural result for a particularly interesting subset of \mathcal{F} .

1. INTRODUCTION AND RESULTS

We study holomorphic mappings between the sphere $\mathbb{S}^2 \subset \mathbb{C}^2$ and the hyperquadric $\mathbb{S}_\varepsilon^3 \subset \mathbb{C}^3$, which for $\varepsilon = \pm 1$ is given by $\mathbb{S}_\pm^3 := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 \pm |z_3|^2 = 1\}$, so that $\mathbb{S}_+^3 = \mathbb{S}^3$ is the sphere in \mathbb{C}^3 . Faran [Far82] classified holomorphic mappings between spheres in \mathbb{C}^2 and \mathbb{C}^3 and Lebl [Leb11] classified mappings sending \mathbb{S}^2 to \mathbb{S}_ε^3 . In [Rei14a] we give a new CR-geometric approach to reprove Faran's and Lebl's results in a unified manner. Let us introduce the following equivalence relation. For $k = 1, 2$ let $H_k : U_k \rightarrow \mathbb{C}^3$ be a holomorphic mapping where U_k is an open and connected neighborhood of $p_k \in \mathbb{S}^2$ and $H_k(U_k \cap \mathbb{S}^2) \subset \mathbb{S}_\varepsilon^3$. We say H_1 is *equivalent* to H_2 if there exist automorphisms ϕ and ϕ' of \mathbb{S}^2 and \mathbb{S}_ε^3 respectively such that $H_2 = \phi' \circ H_1 \circ \phi^{-1}$.

Theorem 1.1 ([Rei14a, Theorem 1.3]). *Let $p \in \mathbb{S}^2$, $U \subset \mathbb{C}^2$ be an open and connected neighborhood of p and $H : U \rightarrow \mathbb{C}^3$ a non-constant holomorphic mapping satisfying $H(U \cap \mathbb{S}^2) \subset \mathbb{S}_\varepsilon^3$. Then H is equivalent to exactly one of the following maps:*

- (i) $H_1^\varepsilon(z, w) = (z, w, 0)$
 - (ii) $H_2^\varepsilon(z, w) = \left(z^2, \frac{(1-\varepsilon+z(1+\varepsilon))w}{\sqrt{2}}, w^2 \right)$
 - (iii) $H_3^\varepsilon(z, w) = \left(z, \frac{(1-\varepsilon+z^2(1+\varepsilon))w}{2z}, \frac{(1-\varepsilon+z(1+\varepsilon))w^2}{2z} \right)$
 - (iv) $H_4^\varepsilon(z, w) = \frac{(4z^3, (3(1-\varepsilon)+(1+3\varepsilon)w^2)w, \sqrt{3}(1-\varepsilon+2(1+\varepsilon)w+(1-\varepsilon)w^2)z)}{1+3\varepsilon+3(1-\varepsilon)w^2}$
- Additionally for $\varepsilon = -1$ we have:
- (v) $H_5(z, w) = \left(\frac{(2+\sqrt{2}z)z}{1+\sqrt{2}z+w}, w, \frac{(1+\sqrt{2}z-w)z}{1+\sqrt{2}z+w} \right)$
 - (vi) $H_6(z, w) = \frac{((1-w)z, 1+w-w^2, (1+w)z)}{1-w-w^2}$
 - (vii) $H_7(z, w) = (1, h(z, w), h(z, w))$ for some non-constant holomorphic function $h : \mathbb{C}^2 \rightarrow \mathbb{C}$

In fact we study holomorphic mappings between the Heisenberg hypersurface $\mathbb{H}^2 \subset \mathbb{C}^2$ and \mathbb{H}_ε^3 , where $\mathbb{H}_+^3 = \mathbb{H}^3$ is the Heisenberg hypersurface in \mathbb{C}^3 . The hypersurfaces \mathbb{H}^2 and \mathbb{H}_ε^3 are

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biholomorphic to \mathbb{S}^2 and \mathbb{S}_ε^3 respectively, except one point, and are given by

$$\mathbb{H}^2 = \{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = |z|^2\}, \quad \mathbb{H}_\varepsilon^3 = \{(z'_1, z'_2, w') \in \mathbb{C}^3 : \operatorname{Im} w' = |z'_1|^2 + \varepsilon |z'_2|^2\}.$$

We denote by \mathcal{F} the class of germs of 2-nondegenerate transversal mappings sending a small piece of \mathbb{H}^2 to \mathbb{H}_ε^3 and is introduced in more details in [Definition 2.5](#) below. This is, in some sense, the most natural and interesting class of mappings when studying holomorphic mappings between \mathbb{H}^2 to \mathbb{H}_ε^3 . From [\[Rei14a\]](#) we know that \mathcal{F} consists of mappings belonging to the orbits of the maps listed in (ii)–(vi) of [Theorem 1.1](#) with respect to the equivalence relation of automorphisms introduced above, after composing with an appropriate Cayley transform. A direct consequence of the above theorem and some intermediate classification result from [\[Rei14a\]](#) is the following topological property of the quotient space of \mathcal{F} modulo automorphisms.

Theorem 1.2. *The quotient space \mathcal{F} / \approx with respect to the equivalence relation of [Theorem 1.1](#) is discrete for $\varepsilon = +1$. This property fails to hold for $\varepsilon = -1$.*

The above result was not known before and shows one major difference between holomorphic mappings from the sphere in \mathbb{C}^2 to the sphere in \mathbb{C}^3 and to the hyperquadric with signature $(2, 1)$ in \mathbb{C}^3 . For a germ of a real-analytic CR-submanifold (M, p) of \mathbb{C}^N we write $\operatorname{Aut}_p(M, p)$ for germs of real-analytic CR-diffeomorphisms fixing p , which we refer to as *isotropies* of (M, p) . Let us denote by $\mathcal{G} := \operatorname{Aut}_0(\mathbb{H}^2, 0) \times \operatorname{Aut}_0(\mathbb{H}_\varepsilon^3, 0)$ the direct product of the groups of isotropies of $(\mathbb{H}^2, 0)$ and $(\mathbb{H}_\varepsilon^3, 0)$ respectively, which we introduce in [Definition 2.3](#) below in more details. We study the action of \mathcal{G} on \mathcal{F} given by $\mathcal{G} \times \mathcal{F} \rightarrow \mathcal{F}$, $(\phi, \phi', H) \mapsto \phi' \circ H \circ \phi^{-1}$. We write $\mathfrak{F} \subset \mathcal{F}$ for the set of maps which have only trivial stabilizers. The action is called *proper* if the associated map $(\phi, \phi', H) \mapsto (H, \phi' \circ H \circ \phi^{-1})$ is a proper map. The following result holds:

Theorem 1.3. *The mapping $N : \mathcal{G} \times \mathfrak{F} \rightarrow \mathfrak{F}$ given by $N(\phi, \phi', H) := \phi' \circ H \circ \phi^{-1}$ is a free and proper left action.*

Based on this result we obtain the following theorem concerning the real-analytic structure of \mathfrak{F} , where $\Pi : \mathfrak{F} \rightarrow \mathfrak{N}$ denotes the normalization map induced by the mapping N and \mathfrak{N} denotes a particular set of representatives of the quotient $\mathfrak{F} / \mathcal{G}$ to be given below in [Lemma 5.1](#):

Theorem 1.4. *If $\varepsilon = +1$ then $\Pi : \mathfrak{F} \rightarrow \mathfrak{F} / \mathcal{G}$ is a real-analytic principal fibre bundle with structure group \mathcal{G} . If $\varepsilon = -1$ then \mathfrak{F} is locally mapped to $\mathcal{G} \times \mathfrak{N}$ via local real-analytic diffeomorphisms. In particular \mathfrak{F} is not a smooth manifold.*

Note that the second part of [Theorem 1.4](#) stands in contrast to the case of $\operatorname{Aut}_p(M, p)$. Assuming some nondegeneracy conditions for certain germs of real-analytic CR-submanifolds (M, p) , such as Levi-nondegeneracy, it is known that $\operatorname{Aut}_p(M, p)$ admits a manifold structure, see [\[BER97\]](#), [\[BER99\]](#), [\[BRWZ04\]](#), [\[Kow05\]](#), [\[KZ05\]](#), [\[LM07\]](#), [\[LMZ08\]](#) and [\[JL13\]](#). To prove [Theorem 1.4](#) we use a real-analytic version of the so-called local slice theorem for free and proper actions. For proper smooth actions of non-compact Lie groups the first proof of the local slice theorem was given in [\[Pal61, 2.2.2 Proposition\]](#). In the real-analytic setting a global slice theorem was proved by [\[HHK96, section VI\]](#) and [\[IK00, Theorem 0.6\]](#). We also obtain the following result about the different topologies we can associate to the quotient space $\mathfrak{F} / \mathcal{G}$:

Theorem 1.5. *The quotient topology τ_Q on $\mathfrak{F} / \mathcal{G} \simeq \mathfrak{N}$ coincides with the induced topology τ_J of \mathfrak{F} , which carries the topology induced by the jet space $J_0^3(\mathbb{H}^2, \mathbb{H}_\varepsilon^3)$.*

We organize this paper as follows: We introduce the necessary notations, tools and results in [section 2](#). In [section 3](#) we study different normal forms with respect to isotropies and in [section 4](#) we investigate the connectedness of \mathcal{F} and discreteness of the quotient space. In the remaining sections we study properties of the action of the group of isotropies on \mathcal{F} , which finally result in some structural and topological information of \mathfrak{F} and $\mathfrak{F} / \mathcal{G}$ respectively in [section 7](#). This article

is partly based on the author's thesis [Rei14b] at the University of Vienna. Some computations are carried out with *Mathematica 7.0.1.0* [Wol08].

2. PRELIMINARIES

Definition 2.1. We fix coordinates $(z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^{n+1}$. For $h : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ a holomorphic function $h(z, w) = \sum_{\alpha, \beta} a_{\alpha\beta} z^\alpha w^\beta$ defined near 0 we write $\bar{h}(\bar{z}, \bar{w}) := \overline{h(z, w)} = \sum_{\alpha, \beta} \bar{a}_{\alpha\beta} \bar{z}^\alpha \bar{w}^\beta$ for the complex conjugate of h . Derivatives of h with respect to z or w we denote by $h_{z^\alpha w^\beta}(0) := \frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial w^\beta} h(0)$. For $n \geq 1$ and a map $H : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n'+1}$ defined near 0 with components $H = (f_1, \dots, f_{n'}, g)$ we write $H_{z^\alpha w^\beta}(0) = (f_{1z^\alpha w^\beta}(0), \dots, f_{n'z^\alpha w^\beta}(0), g_{z^\alpha w^\beta}(0))$.

2.1. Classes of Maps, Automorphisms and Equivalence Relations.

Definition 2.2. We write $\mathcal{H}(p; p') := \{H : (\mathbb{C}^N, p) \rightarrow (\mathbb{C}^{N'}, p') : H \text{ holomorphic}\}$ for the set of germs of holomorphic mappings from (\mathbb{C}^N, p) to $(\mathbb{C}^{N'}, p')$. For $(M, p) \subset \mathbb{C}^N$ and $(M', p') \subset \mathbb{C}^{N'}$ germs of real-analytic hypersurfaces we denote by

$$\mathcal{H}(M, p; M', p') := \{H \in \mathcal{H}(p; p') : H(M \cap U) \subset M' \text{ for some neighborhood } U \text{ of } p\},$$

the set of germs of holomorphic mappings from (M, p) to (M', p') .

Definition 2.3. (i) The collection of germs of locally real-analytic CR-diffeomorphisms of (M, p) we denote by $\text{Aut}(M, p) := \{H : (\mathbb{C}^N, p) \rightarrow \mathbb{C}^N : H \text{ holomorphic}, H(M) \subset M, \det(H'(p)) \neq 0\}$ and the group of isotropies of (M, p) fixing p by $\text{Aut}_p(M, p) := \{H \in \text{Aut}(M, p) : H(p) = p\}$.

(ii) We write $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$, denote the unit sphere in \mathbb{C} by $\mathbb{S}^1 := \{e^{it} : 0 \leq t < 2\pi\}$ and set $\Gamma := \mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1 \times \mathbb{C}$. For an element $\sigma_\gamma \in \text{Aut}_0(\mathbb{H}^2, 0)$ we denote $\gamma = (\lambda, r, u, c) \in \Gamma$ and write:

$$\sigma_\gamma(z, w) := \frac{(\lambda u(z + cw), \lambda^2 w)}{1 - 2i\bar{c}z + (r - i|c|^2)w}. \quad (2.1)$$

(iii) We define for $\sigma = \pm 1$ if $\varepsilon = -1$ and $\sigma = +1$ if $\varepsilon = +1$

$$\mathcal{S}_{\varepsilon, \sigma}^2 := \{a' = (a'_1, a'_2) \in \mathbb{C}^2 : |a'_1|^2 + \varepsilon |a'_2|^2 = \sigma\}, \quad (2.2)$$

and let

$$U' := \begin{pmatrix} u'a'_1 & -\varepsilon u'a'_2 \\ \bar{a}'_2 & \bar{a}'_1 \end{pmatrix}, \quad u' \in \mathbb{S}^1, \quad a' = (a'_1, a'_2) \in \mathcal{S}_{\varepsilon, \sigma}^2. \quad (2.3)$$

We set $\Gamma' := \mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1 \times \mathcal{S}_{\varepsilon, \sigma}^2 \times \mathbb{C}^2$ to denote elements $\sigma'_{\gamma'} \in \text{Aut}_0(\mathbb{H}_\varepsilon^3, 0)$ via $\gamma' = (\lambda', r', u', a', c') \in \Gamma'$, where $c' = (c'_1, c'_2)$:

$$\sigma'_{\gamma'}(z', w') := \frac{(\lambda' U'^t(z' + c'w'), \sigma \lambda'^2 w')}{1 - 2i(\bar{c}'_1 z'_1 + \varepsilon \bar{c}'_2 z'_2) + (r' - i(|c'_1|^2 + \varepsilon |c'_2|^2))w'}. \quad (2.4)$$

(iv) We call elements of $\Gamma \times \Gamma'$ *standard parameters*. If the standard parameters $(\gamma, \gamma') \in \Gamma \times \Gamma'$ are chosen such that $(\sigma_\gamma, \sigma'_{\gamma'}) = (\text{id}_{\mathbb{C}^2}, \text{id}_{\mathbb{C}^3})$, we say the standard parameters are *trivial*.

Definition 2.4. For $G, H \in \mathcal{H}(M, p; M', p')$ we denote the following equivalence relation: $G \sim H : \Leftrightarrow \exists (\phi, \phi') \in \text{Aut}_p(M, p) \times \text{Aut}_{p'}(M', p') : G = \phi' \circ H \circ \phi^{-1}$. The equivalence classes in $\mathcal{H}(M, p; M', p')/\sim$ are denoted by $[F] := \{G \in \mathcal{H}(M, p; M', p') : G \sim F\}$.

In the case where $(p, p') = (0, 0)$ and $(M, M') = (\mathbb{H}^2, \mathbb{H}_\varepsilon^3)$ we call the above relation *isotropic equivalence* and write $O_0(H)$ for the orbit of a map H , called the *isotropic orbit* of H .

2.2. The Class \mathcal{F} , the Normal Form \mathcal{N} and its Classification. In [Rei14a] we introduced the following class of mappings, which are 2-nondegenerate and transversal. These mappings represent the immersive maps, which are not equivalent to the linear embedding, see [Rei14a, Proposition 2.16].

Definition 2.5. For a neighborhood $U \subset \mathbb{C}^2$ of 0 let us denote the set $\mathcal{F}(U)$ of holomorphic mappings $H = (f_1, f_2, g)$ with $H(U \cap \mathbb{H}^2) \subset \mathbb{H}_\varepsilon^3$, which satisfy $H(0) = 0$, $f_{1z}(0)f_{2z^2}(0) - f_{2z}(0)f_{1z^2}(0) \neq 0$ and $g_w(0) > 0$. We denote by \mathcal{F} the set of germs H , such that $H \in \mathcal{F}(U)$ for some neighborhood $U \subset \mathbb{C}^2$ of 0.

Proposition 2.6 ([Rei14a, Proposition 3.1]). *Let $H \in \mathcal{F}$. Then there exist isotropies $(\sigma, \sigma') \in \mathcal{G}$ such that $\hat{H} := \sigma' \circ H \circ \sigma^{-1}$ satisfies $\hat{H}(0) = 0$ and the following conditions:*

$$\begin{array}{lll} \text{(i)} \quad \hat{H}_z(0) = (1, 0, 0) & \text{(iv)} \quad \hat{f}_{2zw}(0) = 0 & \text{(vii)} \quad \text{Re}(\hat{f}_{2z^2w}(0)) = 0 \\ \text{(ii)} \quad \hat{H}_w(0) = (0, 0, 1) & \text{(v)} \quad \hat{f}_{1w^2}(0) = |\hat{f}_{1w^2}(0)| \geq 0 & \\ \text{(iii)} \quad \hat{f}_{2z^2}(0) = 2 & \text{(vi)} \quad \text{Re}(\hat{g}_{w^2}(0)) = 0 & \end{array}$$

A holomorphic mapping of \mathcal{F} satisfying the above conditions is called a normalized mapping. The set of normalized mappings is denoted by \mathcal{N} .

Remark 2.7. A mapping $H \in \mathcal{N}$ necessarily satisfies the following conditions, see [Rei14a, Remark 3.4]:

$$\begin{array}{ll} \text{(i)} \quad H(0) = (0, 0, 0) & \text{(v)} \quad H_{zw}(0) = (\frac{i\varepsilon}{2}, 0, 0) \\ \text{(ii)} \quad H_z(0) = (1, 0, 0) & \text{(vi)} \quad H_{w^2}(0) = (|f_{1w^2}(0)|, f_{2w^2}(0), 0) \\ \text{(iii)} \quad H_w(0) = (0, 0, 1) & \text{(viii)} \quad H_{z^2w}(0) = (4i|f_{1w^2}(0)|, i\text{Im}(f_{2z^2w}(0)), 0) \\ \text{(iv)} \quad H_{z^2}(0) = (0, 2, 0) & \end{array}$$

We classify all mappings belonging to \mathcal{N} in [Rei14a].

Theorem 2.8 ([Rei14a, Theorem 4.1]). *The set \mathcal{N} consists of the following mappings, where $s \geq 0$:*

$$\begin{aligned} G_1^\varepsilon(z, w) &:= \left(2z(2 + i\varepsilon w), 4z^2, 4w \right) / (4 - w^2), \\ G_{2,s}^\varepsilon(z, w) &:= \left(4z - 4\varepsilon sz^2 + i(\varepsilon - s^2)zw + sw^2, 4z^2 + s^2w^2, w(4 - 4\varepsilon sz - i(\varepsilon + s^2)w) \right) \\ &\quad / \left(4 - 4\varepsilon sz - i(\varepsilon + s^2)w - 2iszw - \varepsilon s^2w^2 \right), \\ G_{3,s}^\varepsilon(z, w) &:= \left(256\varepsilon z + 96iszw + 64\varepsilon sw^2 + 64z^3 + 64i\varepsilon sz^2w - 3(3\varepsilon - 16s^2)zw^2 + 4isw^3, \right. \\ &\quad 256\varepsilon z^2 - 16w^2 + 256sz^3 + 16iz^2w - 16\varepsilon szw^2 - i\varepsilon w^3, \\ &\quad \left. w(256\varepsilon - 32iw + 64z^2 - 64i\varepsilon szw - (\varepsilon + 16s^2)w^2) \right) \\ &\quad / \left(256\varepsilon - 32iw + 64z^2 - 192i\varepsilon szw - (17\varepsilon + 144s^2)w^2 + 32i\varepsilon z^2w + 24szw^2 \right. \\ &\quad \left. + iw^3 \right). \end{aligned}$$

Each mapping in \mathcal{N} is not isotropically equivalent to any different mapping in \mathcal{N} .

For $\varepsilon = -1$ we can give the following picture of \mathcal{N} in the s -parameter space according to Theorem 2.8, see [Rei14a, §4] for more details:

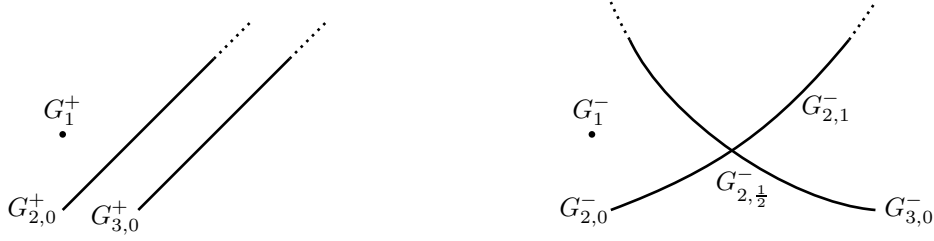


FIGURE 1. \mathcal{N} for $\varepsilon = \pm 1$ in the parameter space

An immediate consequence of the normalization and classification of maps in \mathcal{F} is the following jet determination result.

Corollary 2.9 ([Rei14a, Corollary 4.8]). *Let $U \subset \mathbb{C}^2$ be a neighborhood of 0 and $H : U \rightarrow \mathbb{C}^3$ a holomorphic mapping. We denote the components of H by $H = (f, g) = (f_1, f_2, g)$ and write $j_0(H) := \{j_0^2(H), f_{z^2w}(0)\}$. If for $H_1, H_2 \in \mathcal{F}$ the coefficients belonging to $j_0(H_1)$ and $j_0(H_2)$ coincide, we have $H_1 \equiv H_2$.*

2.3. Associated Topologies. We deal with the following topologies, see e.g. [BER97].

Definition 2.10. For $K \subset \mathbb{C}^N$ a compact neighborhood of $p \in \mathbb{C}^N$ we denote the Frechét space $\mathcal{H}_K(p; p')$ of germs of holomorphic mappings, defined in a neighborhood of K , which map $p \in \mathbb{C}^N$ to $p' \in \mathbb{C}^{N'}$. The topology for $\mathcal{H}_K(p; p')$ is given by uniform convergence on compact sets. We equip $\mathcal{H}(p; p')$ with the inductive limit topology, denoted by τ_C , with respect to Frechét spaces $\mathcal{H}_K(p; p')$, where K is some compact neighborhood of p in \mathbb{C}^N . Then for $H, H_n \in \mathcal{H}(p; p')$ we say that H_n converges to H , if there exists $K \subset \mathbb{C}^N$ a compact neighborhood of p , such that each H_n is holomorphic in a neighborhood of K and H_n converges uniformly to H on K .

For $\mathcal{H}(M, p; M', p') \subset \mathcal{H}(p; p')$ we consider the induced topology of $\mathcal{H}(p; p')$ denoted by τ_C .

Definition 2.11. Let $H : \mathbb{C}^N \rightarrow \mathbb{C}^{N'}$ be a holomorphic mapping defined at $p \in \mathbb{C}^N$ and $\alpha \in \mathbb{N}^N$. We denote by $j_p^k H$ the k -jet of H at p defined as

$$j_p^k H := \left(\frac{\partial^{|\alpha|} H}{\partial Z^\alpha}(p) : |\alpha| \leq k \right).$$

We denote by $J_{p,p'}^k$ the collection of all k -jets at p of germs of mappings from (\mathbb{C}^N, p) to $(\mathbb{C}^{N'}, p')$. We set $J_p^k := J_{p,p}^k$. Let $(M, p) \subset (\mathbb{C}^N, p)$ and $(M', p') \subset (\mathbb{C}^{N'}, p')$ be germs of submanifolds. For $k \in \mathbb{N}$ we denote by $J_q^k(M, p; M', p')$ the space of k -jets of $\mathcal{H}(M, p; M', p')$ at q . We write $J_q^k(M, p) := J_q^k(M, p; M, p)$ and $J_0^k(M; M') := J_0^k(M, 0; M', 0)$. We denote by $G_p^k(M, p) \subset J_p^k(M, p)$ the space of k -jets of $\text{Aut}_p(M, p)$ at p .

Remark 2.12. Note that $J_p^k(M, p; M', p') \subset J_{p,p'}^k$. We identify $J_{p,p'}^k$ with the space of germs of holomorphic polynomial mappings from \mathbb{C}^N to $\mathbb{C}^{N'}$ up to degree k , which map $p \in \mathbb{C}^N$ to $p' \in \mathbb{C}^{N'}$. Thus $J_{p,p'}^k$ can be identified with some \mathbb{C}^K , where $K := N' \binom{N+k}{N}$, such that the topology for $J_{p,p'}^k$, denoted by τ_J , is induced by the natural topology of \mathbb{C}^K . We refer to the topology τ_J as *topology of the jet space*.

Definition 2.13. We say $\mathcal{K} \subset \mathcal{H}(M, p; M', p')$ admits a *jet parametrization* for \mathcal{K} of order k if the following properties hold: There exists a mapping $\Psi : \mathbb{C}^N \times \mathbb{C}^K \supset U \rightarrow \mathbb{C}^{N'}$, with $K = N' \binom{N+k}{N}$ from above and U an open neighborhood of $\{p\} \times J_p^k(M, p; M', p')$, which is holomorphic in the first N variables, real-analytic in the remaining K variables, such that $F(Z) = \Psi(Z, j_p^k F)$, for all $F \in \mathcal{K}$.

Remark 2.14. If $\mathcal{K} \subset \mathcal{H}(M, p; M', p')$ admits a jet parametrization of some order k , then $\tau_{\mathcal{K}} = \tau_J$, which follows from the real-analyticity in the last K variables.

Remark 2.15. Based on [Lam01, Proposition 25, Corollary 26–27] we obtain a jet parametrization of order 4 for $\mathcal{K} = \mathcal{F}$ in [Rei14a, Lemma 4.3] and by Corollary 2.9 we have that $K = K_0 := 15$. Using Theorem 2.8 and the notation from Corollary 2.9 we identify \mathcal{F} with a subset $\mathfrak{J} \subset \mathbb{C}^{K_0}$ given by $\mathfrak{J} := \{j_0(H) : H \in \mathcal{F}\}$ and the topology we use in the sequel for \mathcal{F} is τ_J .

Definition 2.16. Let X be topological spaces, Y a set and $q : X \rightarrow Y$ a surjective mapping. We call the topology on Y induced by q the *quotient topology* τ_Q on Y , where a set $U \subset Y$ is open in Y if $q^{-1}(U)$ is open in X .

3. HOMEOMORPHIC VARIATIONS OF NORMAL FORMS

Definition 3.1. Let \mathcal{H} be a subset of $\mathcal{H}(M, p; M', p')$. A proper subset $\mathcal{K} \subsetneq \mathcal{H}$ is called *normal form for \mathcal{H}* , if for each $[F] \in \mathcal{H}/\sim$, there exists a unique representative $G \in \mathcal{K} \cap [F]$. We denote the mapping which assigns to each $H \in \mathcal{H}$ the representative $G \in \mathcal{K} \cap [H]$ as $\pi : \mathcal{H} \rightarrow \mathcal{K}$. A normal form \mathcal{K} for \mathcal{H} is called *admissible* if $\pi : \mathcal{H} \rightarrow \mathcal{K}$ is continuous.

Remark 3.2. The uniqueness of the representative $F \in \mathcal{K} \cap [F]$ in Definition 3.1 is not a restriction: Assume we have another representative $F \neq G \in \mathcal{K}$ in the class $[F]$, then G is equivalent to F , hence it suffices to choose only one element from the set of all representatives which belong to $\mathcal{K} \cap [F]$. If there exists an admissible normal form \mathcal{K} for \mathcal{H} we observe that in each orbit of any not necessarily admissible normal form \mathcal{K}' for \mathcal{H} , there exists an element of $\mathcal{K} \cap \mathcal{K}'$.

Theorem 3.3. Let \mathcal{N}' be an admissible normal form for \mathcal{F} . Then \mathcal{N}' is homeomorphic to \mathcal{N} , where we equip \mathcal{N}' and \mathcal{N} with τ_J .

Proof. Let us denote by $\pi' : \mathcal{F} \rightarrow \mathcal{N}'$ the continuous mapping as in Definition 3.1. We note that the class \mathcal{N} introduced in Proposition 2.6 is an admissible normal form for \mathcal{F} as in Definition 2.5. To see this we need to inspect the proof of Proposition 2.6 given in [Rei14a, Proposition 3.1]. If we equip \mathcal{F} with τ_J we obtain that for the isotropies $(\sigma, \sigma') \in \mathcal{G}$ deduced in the proof, the mapping $\pi : \mathcal{F} \rightarrow \mathcal{N}, H \mapsto \sigma' \circ H \circ \sigma^{-1}$ is continuous, since the isotropies depend real-analytically on $j_0^3(H)$. Hence we have the following diagram:

$$\begin{array}{ccc} & \mathcal{F} & \\ \pi' \nearrow & & \nwarrow \pi \\ \mathcal{N}' & \xrightarrow{\quad \psi \quad} & \mathcal{N} \\ \text{incl}' \nearrow & & \nwarrow \text{incl} \end{array}$$

FIGURE 2. Diagram for admissible normal forms

The mapping $\text{incl}' : \mathcal{N}' \rightarrow \mathcal{F}$ is the inclusion mapping, which is given by $\text{incl}'(H) := H$ for all $H \in \mathcal{N}'$ and similar for $\text{incl} : \mathcal{N} \rightarrow \mathcal{F}$. The map $\psi : \mathcal{N}' \rightarrow \mathcal{N}$ is given by $\psi(H) := F$ for $H \in \mathcal{N}'$ and $F \in \mathcal{N} \cap [H]$. Since \mathcal{N}' and \mathcal{N} are normal forms, we obtain that ψ is a bijective mapping. Further since $\psi = \pi \circ \text{incl}'$ and $\psi^{-1} = \pi' \circ \text{incl}$ are compositions of continuous mappings, we obtain that ψ is a homeomorphism. \square

Example 3.4. Starting with \mathcal{N} we can construct different admissible normal forms \mathcal{N}' as follows: We fix a pair of isotropies $(\phi_0, \phi'_0) \in \mathcal{G}$ and consider the isotropies $(\hat{\phi}, \hat{\phi}') \in \mathcal{G}$ from

Proposition 2.6, such that $\pi : \mathcal{F} \rightarrow \mathcal{N}$ is given by $\pi(H) := \hat{\phi}' \circ H \circ \hat{\phi}^{-1}$, denoted by \hat{H} . We define $\phi := \phi_0 \circ \hat{\phi}$ and $\phi' := \phi'_0 \circ \hat{\phi}'$, to obtain for $F \in \mathcal{F}$,

$$\phi' \circ F \circ \phi^{-1} = \phi'_0 \circ \hat{\phi}' \circ F \circ \hat{\phi}^{-1} \circ \phi_0^{-1} = \phi'_0 \circ \hat{F} \circ \phi_0^{-1},$$

where $\hat{F} \in \mathcal{N}$. We define $\mathcal{N}' := \{\phi'_0 \circ \hat{F} \circ \phi_0^{-1} : \hat{F} \in \mathcal{N}\}$. Since $\hat{\phi}$ and $\hat{\phi}'$ depend continuously on $F \in \mathcal{F}$, the mapping $\pi' : \mathcal{F} \rightarrow \mathcal{N}'$ given by $\pi'(F) := \phi' \circ F \circ \phi^{-1}$ is continuous, which means that \mathcal{N}' is an admissible normal form.

4. A TOPOLOGICAL PROPERTY OF THE QUOTIENT SPACE OF \mathcal{F}

Lemma 4.1. *The class \mathcal{F} consists of $\frac{5+\varepsilon}{2}$ connected components.*

Proof. We denote by $\pi : \mathcal{F} \rightarrow \mathcal{N}$ the continuous map, which takes $F \in \mathcal{F}$ to $\hat{F} \in \mathcal{N}$ according to Proposition 2.6. For $k = 2, 3$ we set $C_k := \{G_{k,s}^\varepsilon : s \geq 0\}$ and $\hat{\mathcal{N}} := C_2 \cup C_3$. The space of standard parameters $\Gamma \times \Gamma'$ is path-connected, since for mappings in \mathcal{F} and $\varepsilon = -1$ we only consider isotropies as in (2.4) with $\sigma = +1$. Thus for any $H \in \mathcal{N}$ the isotropic orbit $O_0(H)$ is path-connected. First we treat the case $\varepsilon = -1$. We observe that $\hat{\mathcal{F}} = \bigcup_{H \in \hat{\mathcal{N}}} O_0(H)$ is path-connected. If \mathcal{F} would be connected then $\pi(\mathcal{F}) = \mathcal{N}$ is connected, which is not the possible, since \mathcal{N} consists of 2 connected components G_1^- and $\hat{\mathcal{N}}$. Thus \mathcal{F} has 2 connected components $O_0(G_1^-)$ and $\hat{\mathcal{F}}$. For $\varepsilon = +1$ we note that the set $O_0(C_k) := \bigcup_{H \in C_k} O_0(H)$ for $k = 2, 3$ is path-connected and \mathcal{N} consists of 3 connected components. Thus \mathcal{F} admits at most 3 connected components. \mathcal{F} is not connected since then $\pi(\mathcal{F}) = \mathcal{N}$ would be connected. If \mathcal{F} consists of 2 connected components $\mathcal{F}_1, \mathcal{F}_2$, such that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, we need to distinguish several cases. Either $\mathcal{F}_1 = O_0(G_1^+) \cup O_0(C_k)$, and $\mathcal{F}_2 = O_0(C_\ell)$, where $k \neq \ell$ and $k, \ell \in \{2, 3\}$ or $\mathcal{F}_1 = O_0(C_2) \cup O_0(C_3)$ and $\mathcal{F}_2 = O_0(G_1^+)$. In all cases we have by the continuity of π , that $\pi(\mathcal{F}_1)$ is connected, which is not possible. \square

Proof of Theorem 1.2. The set $X := \mathcal{F} / \sim$ consists of elements denoted by $[F]$ for $F \in \mathcal{F}$. We equip X with the quotient topology such that the canonical projection $\pi : \mathcal{F} \rightarrow X$ is continuous. For $\varepsilon = +1$ we have $X = \{G_1^+, G_{2,0}^+, G_{3,0}^+\}$ by our classification. By Lemma 4.1 we obtain that $\pi^{-1}(H)$ for $H \in X$ is a connected component of \mathcal{F} , hence open. Thus X carries the discrete topology. To prove the statement for $\varepsilon = -1$ we write $H_0 := G_{2,1/2}^- \in \mathcal{N}$ and $H_1 := G_{3,0}^- \in \mathcal{N}$. For $k = 0, 1$ let $U_k \in X$ be an open neighborhood of $[H_k]$, then $V_k := \pi^{-1}(U_k)$ is an open neighborhood of the orbit of H_k in \mathcal{F} . According to our classification there exists a sequence $(G_n)_{n \in \mathbb{N}}$ of mappings in \mathcal{F} , where each $G_n \in [H_1]$ and $G_n \rightarrow H_0$ in \mathcal{F} as $n \rightarrow \infty$. Thus there exists $N \in \mathbb{N}$ such that $G_n \in V_0 \cap V_1$ for all $n \geq N$, which shows $[H_1] \in U_0 \cap U_1$ and completes the proof. \square

5. ISOTROPIC STABILIZER

Lemma 5.1. *We set $\mathfrak{N} := \mathcal{N} \setminus \{G_1^\varepsilon, G_{2,0}^\varepsilon, G_{3,0}^\varepsilon\}$ and $\mathfrak{F} := \bigcup_{H \in \mathfrak{N}} O_0(H)$. The isotropic stabilizer $\text{stab}_0(H) := \{(\phi, \phi') \in \mathcal{G} : \phi' \circ H \circ \phi^{-1} = H\}$ of H is trivial for $H \in \mathfrak{N}$. Furthermore we have $\text{stab}_0(G_1^\varepsilon) = \text{stab}_0(G_{2,0}^\varepsilon)$ is homeomorphic to \mathbb{S}^1 and $\text{stab}_0(G_{3,0}^\varepsilon)$ is homeomorphic to \mathbb{Z}_2 .*

Proof. We let $H = (f, g) = (f_1, f_2, g) \in \mathcal{N}$ satisfy the conditions we collected in Remark 2.7. We write $s := 2|f_{1w^2}(0)| \geq 0, x := f_{2w^2}(0) \in \mathbb{C}$ and $y := \text{Im}(f_{2z^2w}(0)) \in \mathbb{R}$. By Corollary 2.9 we only need to consider coefficients in $j_0(H)$. We let $(\sigma, \sigma') \in \mathcal{G}$ with the notation from (2.1), (2.3) and (2.4) respectively and consider the equation

$$\sigma' \circ H \circ \sigma^{-1} = H, \tag{5.1}$$

where we parametrize σ^{-1} as in (2.1). The coefficients of order 1, which are $f_z(0)$ and $H_w(0)$, are given by $U'^t(u\lambda\lambda', 0) = (1, 0)$ and $U'^t(uc + \lambda c'_1, \lambda c'_2, \lambda\lambda') = (0, 0, 1)$. These equations imply $\lambda' = 1/\lambda$, $a'_2 = c'_2 = 0$, $a'_1 = 1/(uu')$ and $c'_1 = -uc/\lambda$. Assuming these standard parameters we consider the coefficients of order 2, which are $f_{z^2}(0)$, $H_{zw}(0)$ and $H_{w^2}(0)$, given by:

$$(0, 2u'u^3\lambda) = (0, 2), \quad (5.2)$$

$$(-r - \lambda^2 r' + i\varepsilon \lambda^2/2, 2u'u^3\lambda c, 0) = (i\varepsilon/2, 0, 0), \quad (5.3)$$

$$(\lambda^2(\lambda s + i\varepsilon uc)/u, uu'\lambda(\lambda^2 x + 2u^2 c^2), -2(r + \lambda^2 r')) = (s, x, 0). \quad (5.4)$$

The second component of (5.3) implies $c = 0$. If we assume this value for c we obtain for the third order terms $f_{z^2 w}(0)$ the following equation:

$$(2iu\lambda^3 s, u'u^3\lambda(-4r - 2\lambda^2 r' + i\lambda^2 y)) = (4is, iy). \quad (5.5)$$

The second component of (5.2) shows $\lambda = 1$. Furthermore we obtain from the third component of (5.4) that $r' = -r$ and since from the second component of (5.2) we get $u'u^3 = 1$, which uniquely determines u' , we obtain from the second component of (5.5) that $r = 0$. The remaining equation from the first component of (5.4), which comes from the coefficient $f_{1w^2}(0)$, is $s/u = s$. If $s > 0$ we obtain that $u = 1$ and hence all standard parameters are trivial, which proves the first claim of the lemma. If $s = 0$, then $H \in \{G_1^\varepsilon, G_{2,0}^\varepsilon, G_{3,0}^\varepsilon\}$, since these maps are precisely the one satisfying $f_{1w^2}(0) = 0$ in the list of mappings from Theorem 2.8. It is easy to check that the isotropic stabilizers of the maps G_1^ε and $G_{2,0}^\varepsilon$ are generated by the isotropies $(\sigma(z, w), \sigma'(z'_1, z'_2, w')) = (uz, w, z'_1/u, z'_2/u^2, w')$ with $|u| = 1$. If we consider $G_{3,0}^\varepsilon$ in (5.1), then we obtain that $(\sigma(z, w), \sigma'(z'_1, z'_2, w')) = (\delta z, w, \delta z'_1, z'_2, w')$, where $\delta = \pm 1$, are the only elements of $\text{stab}_0(G_{3,0}^\varepsilon)$, which proves the last claim of the lemma. \square

6. PROPERTIES OF THE GROUP ACTION

Lemma 6.1 ([tD87, Proposition 3.20]). *Let G be a topological group acting freely on a topological space X via the action $\alpha : G \times X \rightarrow X$. Then the following statements are equivalent:*

- (i) *G acts properly.*
- (ii) *Let $\alpha' : G \times X \rightarrow X \times X$ be given by $\alpha'(g, x) := (x, \alpha(g, x))$. The image $C \subset X \times X$ of α' is closed and the map $\varphi_\alpha : C \rightarrow G$, given by $\varphi_\alpha(x, \alpha(g, x)) := g$ is continuous.*

Remark 6.2. For $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ a germ of a holomorphic mapping, for which we assume that $F \in \mathcal{F}$ and the jet $j_0(F) \subset j_0^3 F$ is of the form as in Remark 2.7, we write $F = (f^1, f^2, f^3)$ for the components and denote derivatives of F at 0 by $f_{\ell m}^k := f_{z^\ell w^m}^k(0)$.

Lemma 6.3. *For $n \in \mathbb{N}$ we let $H_n, H \in \mathfrak{H}$ and $(\phi_n, \phi'_n) \in \mathcal{G}$ such that $\phi'_n \circ H_n \circ \phi_n^{-1} \rightarrow H$, then $(\phi_n, \phi'_n) \rightarrow (\text{id}_{\mathbb{C}^2}, \text{id}_{\mathbb{C}^3})$ and $H_n \rightarrow H$ as $n \rightarrow \infty$.*

Proof. We assume for $H_n = (h_n^1, h_n^2, h_n^3)$ and $H = (h^1, h^2, h^3)$ to be given as in Remark 6.2, where in H_n the coefficients depend on $n \in \mathbb{N}$. We write $s_n := 2|h_{n02}^1| \in \mathbb{R}^+$, $x_n := h_{n02}^2 \in \mathbb{C}$ and $y_n := \text{Im}(h_{n21}^2)$. To each $(\phi_n, \phi'_n) \in \mathcal{G}$ we associate $(\gamma_n, \gamma'_n) \in \Gamma \times \Gamma'$ respectively, where we use the notation for the parametrization of \mathcal{G} from (2.1) and (2.4). According to Theorem 2.8 we have that H_n depends on $s_n > 0$. Let us denote $\Xi := \Gamma \times \Gamma' \times \mathbb{R}^+$ and write $\xi_n = (\gamma_n, \gamma'_n, s_n) \in \Xi$. We define $\Psi_n := \phi'_n \circ H_n \circ \phi_n^{-1}$, which depends on $\xi_n \in \Xi$. For components of Ψ_n , we write $\Psi_n = (\psi_n^1, \psi_n^2, \psi_n^3)$ and $\psi_n = (\psi_n^1, \psi_n^2)$. Limits are always considered when $n \rightarrow \infty$. We start with the first order terms of Ψ_n . We let U'_n be the 2×2 -matrix from (2.3) with entries u'_n, a'_{1n} and a'_{2n} instead of u', a'_1 and a'_2 , then we have

$$\psi_{nz}(0) = \lambda_n \lambda'_n U'_n{}^t(u_n, 0), \quad (6.1)$$

$$\Psi_{nw}(0) = \lambda_n \lambda'_n \left(U'_n{}^t(u_n c_n + \lambda_n c'_{1n}, \lambda_n c'_{2n}), \lambda_n \lambda'_n \right). \quad (6.2)$$

Since $\psi_{nw}^3(0) \rightarrow 1$ we obtain $\lambda_n \lambda'_n \rightarrow 1$, which implies if we consider (6.1), since $\psi_{nz}(0) \rightarrow (1, 0)$, that $u_n u'_n a'_{1n} \rightarrow 1$ and $a'_{2n} \rightarrow 0$. Because $a'_n = (a'_{1n}, a'_{2n}) \in \mathcal{S}_{\varepsilon, \sigma}^2$ from (2.2), we have $|a'_{1n}| \rightarrow 1$. If we consider the first two components in (6.2) we obtain from $\psi_{nw}(0) \rightarrow (0, 0)$ and $(|a'_{1n}|, |a'_{2n}|) \rightarrow (1, 0)$ that $u_n c_n + \lambda_n c'_{1n} \rightarrow 0$ and $c'_{2n} \rightarrow 0$. Next we consider the second order terms of Ψ_n to obtain

$$\psi_{nz^2}(0) = 2u_n \lambda_n \lambda'_n U_n'^t (2i(\bar{c}_n + u_n \lambda_n \bar{c}'_{1n}), u_n \lambda_n), \quad (6.3)$$

where the left-hand side of (6.3), $\psi_{nz^2}(0)$, must converge to $(0, 2)$. After applying $U_n'^{-1}$ we rewrite the second components of (6.3) as

$$2u_n^2 \lambda_n^2 \lambda'_n = a'_{1n} \left(-\bar{a}'_{2n} \psi_{nz^2}^1(0) / (u'_n a'_{1n}) + \psi_{nz^2}^2(0) \right), \quad (6.4)$$

where since $(|a'_{1n}|, |a'_{2n}|) \rightarrow (1, 0)$ the absolute value of the right-hand side of (6.4) converges to 2. Taking the absolute value of the left-hand side of (6.4) implies $\lambda_n \rightarrow 1$, which together with $\lambda_n \lambda'_n \rightarrow 1$ shows $\lambda'_n \rightarrow 1$. Next we consider

$$\psi_{nzw}(0) = \frac{i}{2} \lambda_n \lambda'_n U_n'^t (T_1(\gamma_n, \gamma'_n), 4\lambda_n (c'_{2n} (\bar{c}_n + u_n \lambda_n \bar{c}'_{1n}) - i u_n^2 c_n)), \quad (6.5)$$

where the real-analytic function $T_1 : \Gamma \times \Gamma' \rightarrow \mathbb{C}$ does not depend on $a'_n \in \mathcal{S}_{\varepsilon, \sigma}^2$ and u'_n . The left-hand side of (6.5) has to converge to $(i\varepsilon/2, 0)$ and we rewrite the second component of (6.5) as

$$4\lambda_n (c'_{2n} (\bar{c}_n + u_n \lambda_n \bar{c}'_{1n}) - i u_n^2 c_n) = -2i \left(-\bar{a}'_{2n} \psi_{nzw}^1(0) + u'_n a'_{1n} \psi_{nzw}^2(0) \right) / (\lambda_n \lambda'_n u'_n). \quad (6.6)$$

Taking the limit, we know since $(|a'_{1n}|, |a'_{2n}|) \rightarrow (1, 0)$ and $(\lambda_n, \lambda'_n) \rightarrow (1, 1)$, that the right-hand side of (6.6) converges to 0 and if we also use $u_n c_n + \lambda_n c'_{1n} \rightarrow 0$ we obtain that $c_n \rightarrow 0$, such that $c'_{1n} \rightarrow 0$. Next we compute

$$\psi_{nw^2}^3(0) = 2\lambda_n^2 \lambda'_n{}^2 \left(-(r_n + \lambda_n^2 r'_n) + i \left(c_n \bar{c}_n + \varepsilon \lambda_n^2 c'_{2n} \bar{c}_{2n'} + \lambda_n \bar{c}'_{1n} (2u_n c_n + \lambda_n c'_{1n}) \right) \right). \quad (6.7)$$

We let $n \rightarrow \infty$ and take all the previously obtained limits of the sequences $c'_n = (c'_{1n}, c'_{2n}) \in \mathbb{C}^2$, c_n and λ_n, λ'_n , then we have since $\psi_{nw^2}^3(0) \rightarrow 0$, that $r_n + \lambda_n^2 r'_n \rightarrow 0$. Next we compute

$$\psi_{nw^2}(0) = \lambda_n \lambda'_n U_n'^t (\lambda_n^3 s_n + T_2(\gamma_n, \gamma'_n), \lambda_n^3 x_n + T_3(\gamma_n, \gamma'_n)), \quad (6.8)$$

where $T_2, T_3 : \Gamma \times \Gamma' \rightarrow \mathbb{C}$ are real-analytic functions and T_2 is given by

$$\begin{aligned} T_2(\gamma_n, \gamma'_n) &= 2(u_n c_n + c'_{1n} \lambda_n) (i |c_n|^2 - r_n - \lambda_n^2 r'_n) + 2i \lambda_n \bar{c}'_{1n} (u_n c_n + \lambda_n c'_{1n}) (2u_n c_n + \lambda_n c'_{1n}) \\ &\quad + i \varepsilon \lambda_n^2 (u_n c_n (1 + 2|c'_{2n}|^2) + 2\lambda_n c'_{1n} |c'_{2n}|), \end{aligned}$$

such that $T_2(\gamma_n, \gamma'_n) \rightarrow 0$. Then the first component of (6.8) becomes

$$\lambda_n^3 s_n + T_2(\gamma_n, \gamma'_n) = \left(\bar{a}'_{1n} \psi_{nw^2}^1(0) + \varepsilon u'_n a'_{2n} \psi_{nw^2}^2(0) \right) / (\lambda_n \lambda'_n u'_n). \quad (6.9)$$

Since $(\psi_{nw^2}^1(0), \psi_{nw^2}^2(0)) \rightarrow (2|h_{02}^1|, h_{02}^2) \in \mathbb{R}^+ \times \mathbb{C}$ we obtain that $\bar{a}'_{1n}/u'_n \rightarrow 1$ and $s_n \rightarrow 2|h_{02}^1|$. Then $u_n u'_n a'_{1n} \rightarrow 1$ implies that $u_n \rightarrow 1$ and further inspection of (6.4) gives $u_n^2/a'_{1n} \rightarrow 1$, which shows $a'_{1n} \rightarrow 1$ and $u'_n \rightarrow 1$. Finally we consider

$$\psi_{nz^2w}(0) = \lambda_n \lambda'_n U_n' \left(\begin{aligned} &-4i u_n^2 \lambda_n^3 s_n + T_4(\gamma_n, \gamma'_n) \\ &-2\varepsilon u_n^2 \lambda_n (2r_n + \lambda_n^2 r'_n) + i \varepsilon u_n^2 \lambda_n^3 y_n + 6u_n^3 \lambda_n^2 c_n s_n + T_5(\gamma_n, \gamma'_n) \end{aligned} \right), \quad (6.10)$$

where $T_4, T_5 : \Gamma \times \Gamma' \rightarrow \mathbb{C}$ are real-analytic functions and T_5 is given by

$$T_5(\gamma_n, \gamma'_n) = 2i\varepsilon\lambda_n \left(4i\bar{c}_n c'_{2n}(\bar{c}_n + 2u_n\lambda_n \bar{c}'_{1n}) + 2c_n u_n^2(5\bar{c}_n + 3u_n\lambda_n \bar{c}'_{1n}) \right. \\ \left. + u_n^2\lambda_n^2(|c'_{1n}|^2 + 3\varepsilon|c'_{2n}|^2 + 4i\bar{c}'_{1n}c'_{2n}) \right),$$

hence $T_5(\gamma_n, \gamma'_n) \rightarrow 0$. Since $(\psi_{nz^2w}^1(0), \psi_{nz^2w}^2(0)) \rightarrow (2i|h_{02}^1|, i h_{21}^2) \in i\mathbb{R} \times i\mathbb{R}$ we obtain if we consider the real part of the second component of (6.10), that $2r_n + r'_n \rightarrow 0$, which together with $r_n + \lambda_n^2 r'_n \rightarrow 0$ shows $(r_n, r'_n) \rightarrow (0, 0)$. To sum up we obtain $(\phi_n, \phi'_n) \rightarrow (\text{id}_{\mathbb{C}^2}, \text{id}_{\mathbb{C}^3})$, which completes the proof. \square

Proof of Theorem 1.3. First we observe that N is a continuous map from $\mathcal{G} \times \mathfrak{F}$ to \mathfrak{F} , since the image of N consists of rational mappings, which depend real-analytically on the jets of the isotropies and the mapping. By construction N is a left action and Lemma 5.1 shows that N restricted to \mathfrak{N} is a free action. Next we assume the general case $H \in \mathfrak{F}$ and consider the equation $\phi' \circ H \circ \phi^{-1} = H$ for $(\phi, \phi') \in \mathcal{G}$. We can write $H = \hat{\phi}' \circ \hat{H} \circ \hat{\phi}^{-1}$, where $\hat{H} \in \mathfrak{N}$ and $(\hat{\phi}, \hat{\phi}') \in \mathcal{G}$ are unique according to Lemma 5.1. After setting $(\psi, \psi') = (\hat{\phi}^{-1} \circ \phi \circ \hat{\phi}, \hat{\phi}'^{-1} \circ \phi' \circ \hat{\phi}')$ we rewrite $\phi' \circ H \circ \phi^{-1} = H$ as $\psi' \circ \hat{H} \circ \psi^{-1} = \hat{H}$. Since N acts freely on \mathfrak{N} we obtain that $(\psi, \psi') = (\text{id}_{\mathbb{C}^2}, \text{id}_{\mathbb{C}^3})$ and the freeness of the action. To show the properness of N we prove (ii) of Lemma 6.1. We let the mapping $N' : \mathcal{G} \times \mathfrak{F} \rightarrow \mathfrak{F} \times \mathfrak{F}$ be given by $N'(\phi, \phi', H) := (H, N(\phi, \phi', H))$. Then we know from Proposition 2.6 that the image C of N' agrees with $\mathfrak{F} \times \mathfrak{F}$, which is closed in $\mathfrak{F} \times \mathfrak{F}$. Next we let the mapping $\varphi_N : C \rightarrow \mathcal{G}$ be given by $\varphi_N(H, N(\phi, \phi', H)) := (\phi, \phi')$. To show the continuity of φ_N we let $(H_n)_{n \in \mathbb{N}} \in \mathfrak{F}$ be a sequence of mappings with

$$H_n \rightarrow H \in \mathfrak{F}, \quad \text{and} \quad \phi'_n \circ H_n \circ \phi_n^{-1} \rightarrow \check{H} \in \mathfrak{F}. \quad (6.11)$$

Using Proposition 2.6 we assume w.l.o.g. $H \in \mathfrak{N}$. Moreover by Proposition 2.6 we write $\check{H} = \phi' \circ \hat{H} \circ \phi^{-1}$ for $\hat{H} \in \mathfrak{N}$. Then we need to conclude that $(\phi_n, \phi'_n) \rightarrow (\phi, \phi')$ and $H = \hat{H}$, which implies the continuity of φ_N . For each $n \in \mathbb{N}$ we write $H_n = \hat{\phi}'_n \circ \hat{H}_n \circ \hat{\phi}_n^{-1}$, where $\hat{H}_n \in \mathfrak{N}$. If we substitute the above representations of H_n and \check{H} into (6.11) we obtain $\varphi'_n \circ \hat{H}_n \circ \varphi_n^{-1} \rightarrow \hat{H} \in \mathfrak{N}$, where $(\varphi_n, \varphi'_n) = (\phi^{-1} \circ \phi_n \circ \hat{\phi}_n, \phi'^{-1} \circ \phi'_n \circ \hat{\phi}'_n)$. By Lemma 6.3 we have $(\varphi_n, \varphi'_n) \rightarrow (\text{id}_{\mathbb{C}^2}, \text{id}_{\mathbb{C}^3})$. Again applying Lemma 6.3 to $H_n \rightarrow H \in \mathfrak{N}$ shows $(\hat{\phi}_n, \hat{\phi}'_n) \rightarrow (\text{id}_{\mathbb{C}^2}, \text{id}_{\mathbb{C}^3})$ to obtain $(\phi_n, \phi'_n) \rightarrow (\phi, \phi')$ as required. \square

7. ON THE REAL-ANALYTIC STRUCTURE OF \mathfrak{F}

Lemma 7.1. *Let $\Pi : \mathfrak{F} \rightarrow \mathfrak{N}$ be given by $\Pi(H) := \phi' \circ H \circ \phi^{-1}$, where $(\phi, \phi') \in \mathcal{G}$ are the unique isotropies according to Proposition 2.6 and Lemma 5.1. For $k = 2, 3$ we write $M_{k,\varepsilon} := \{\Pi^{-1}(G_{k,s}^\varepsilon) : s > 0\}$. Then $M_{k,\varepsilon}$ is a real-analytic real submanifold of \mathfrak{F} of real dimension 16.*

Proof of Lemma 7.1. For fixed $k = 2, 3, s > 0$ and $\delta > 0$ we write $G_{\delta,s}^\varepsilon := \{G_{k,t}^\varepsilon : t \in B_\delta(s) \cap \mathbb{R}^+\}$, where $B_\delta(s) := \{t \in \mathbb{R}^+ : |t - s| < \delta\}$. To prove the lemma we show that for every $s_0 \in \mathbb{R}^+$ and sufficiently small $\delta_0 > 0$ there exists a locally real-analytic parametrization for $M := \Pi^{-1}(G_{\delta_0,s_0}^\varepsilon)$. As noted in Remark 2.15 we identify \mathcal{F} with the set $\mathfrak{F} \subset \mathbb{C}^{K_0}$. Theorem 2.8 implies that for each $H \in M$ there exist $(\phi, \phi') \in \mathcal{G}, k \in \{2, 3\}$ and $s_1 \in B_{\delta_0}(s_0) \cap \mathbb{R}^+$, such that $H = \phi' \circ G_{k,s_1}^\varepsilon \circ \phi^{-1}$. This fact is used to describe M locally via parametrizations as follows: For $s > 0$ sufficiently near s_0 let F_s be a mapping as in Remark 6.2, which depends real-analytically on $s := 2|f_{02}^1|$. For the remaining coefficients in $j_0(F_s)$ we write $x := f_{02}^2$ and $y := \text{Im}(f_{21}^2)$, where we suppress the dependence on s notationally. We use the real version of the notation for the parametrization of \mathcal{G} as in (2.1) and (2.4). Here we denote the set of real parameters of $\text{Aut}_0(\mathbb{H}^2, 0)$ by Γ and of

$\text{Aut}_0(\mathbb{H}_\varepsilon^3, 0)$ by Γ' . Let us denote $\Xi := \Gamma \times \Gamma' \times \mathbb{R}^+ \subset \mathbb{R}^{N_0}$, where $N_0 := 16$. For $\xi = (\gamma, \gamma', s) \in \Xi$ we define the mapping

$$\Psi : \Xi \rightarrow \mathfrak{J}, \quad \Psi(\xi) := j_0(\phi'_{\gamma'} \circ F_s \circ \phi_\gamma^{-1}), \quad (7.1)$$

where we use the notation as in (2.1) and (2.4) for ϕ_γ and $\phi'_{\gamma'}$ respectively and suppress the dependence on ε . We set $\check{\Psi}(z, w) := (\phi'_{\gamma'} \circ F_s \circ \phi_\gamma^{-1})(z, w)$ with components $\check{\Psi} = (\check{\psi}^1, \check{\psi}^2, \check{\psi}^3)$ and $\check{\psi} := (\check{\psi}^1, \check{\psi}^2)$. The holomorphic mapping $\check{\Psi}$ is defined in a small neighborhood $U \subset \mathbb{C}^2$ of 0 and satisfies $\check{\Psi}(\mathbb{H}^2 \cap U) \subset \mathbb{H}_\varepsilon^3$. By Theorem 2.8 and the real-analytic dependence of the isotropies on the standard parameters, we note that Ψ and $\check{\Psi}$ are real-analytic in $\xi \in \Xi$. We assume w.l.o.g. that ξ_0 is chosen in such a way that $(\phi_\gamma, \phi'_{\gamma'}) = (\text{id}_{\mathbb{C}^2}, \text{id}_{\mathbb{C}^3})$. Consequently we write $O(2)$ for terms involving standard parameters of the isotropies which vanish to second order at ξ_0 . Moreover since we only consider $a'_1 \in \mathbb{C}$ near 1 and $a' = (a'_1, a'_2) \in \mathcal{S}_{\varepsilon, \sigma}^2$ from (2.2), we substitute $\bar{a}'_1 = (1 - \varepsilon |a'_2|^2)/a'_1$ into Ψ , which is then given by the following expressions:

$$\begin{aligned} \check{\psi}_z(0) &= (uu' \lambda \lambda' a'_1, u \lambda \lambda' \bar{a}'_2), \\ \check{\Psi}_w(0) &= \left(u' \lambda \lambda' a'_1 (uc + \lambda c'_1), \lambda^2 \lambda' c'_2 / a'_1, \lambda^2 \lambda'^2 \right) + O(2), \\ \check{\psi}_{z^2}(0) &= \left(2i uu' \lambda \lambda' (i\varepsilon u \lambda a'_2 + 2(\bar{c} + u \lambda \bar{c}'_1) a'_1), 2u^2 \lambda^2 \lambda' / a'_1 \right) + O(2), \\ \check{\Psi}_{zw}(0) &= \left(-\frac{1}{2} uu' \lambda \lambda' a'_1 (2(r + \lambda^2 r') - i\varepsilon \lambda^2), u \lambda^2 \lambda' \left(\frac{i\varepsilon}{2} \lambda \bar{a}'_2 + \frac{2uc}{a'_1} \right), 2i \lambda^2 \lambda'^2 (\bar{c} + u \lambda \bar{c}'_1) \right) \\ &\quad + O(2), \\ \check{\Psi}_{w^2}(0) &= \left(u' \lambda^3 \lambda' (a'_1 (i\varepsilon uc + \lambda s) - \varepsilon \lambda a'_2 x), \lambda^4 \lambda' (x/a'_1 + \bar{a}'_2 s), -2\lambda^2 \lambda'^2 (r + \lambda^2 r') \right) + O(2), \\ \check{\psi}_{z^2 w}(0) &= \left(-uu' \lambda^3 \lambda' \left(4a'_1 (-i u \lambda s + \varepsilon (\bar{c} + u \lambda \bar{c}'_1)) + i\varepsilon u \lambda a'_2 y \right), \right. \\ &\quad \left. u^2 \lambda^2 \lambda' \left((-2(2r + \lambda^2 r') + 6\varepsilon u \lambda cs + i \lambda^2 y) / a'_1 + 2i \lambda^2 \bar{a}'_2 s \right) \right) + O(2). \end{aligned}$$

In a first step we show that for given $\xi_0 \in \Xi$ the Jacobian of Ψ with respect to ξ evaluated at ξ_0 , denoted by $\Psi_\xi(\xi_0)$, is of full rank N_0 . But instead of considering the real equations of Ψ , we conjugate Ψ and compute the Jacobian of the system $\Phi := (\Psi, \bar{\Psi}) \in \mathbb{C}^{2K_0}$, with respect to $\xi = (u, \lambda, c, r, u', a'_1, a'_2, \lambda', c'_1, c'_2, r', s; \bar{c}, \bar{a}'_2, \bar{c}'_1, \bar{c}'_2) \in \mathbb{C}^{N_0}$ and evaluate at

$$\xi_0 = (1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, s_0; 0, 0, 0, 0) \in \mathbb{R}^{N_0}, \quad (7.2)$$

denoted by $\Phi_\xi(\xi_0)$. We bring the transpose of $\Phi_\xi(\xi_0)$ into echelon form, where we denote the resulting matrix by $\varphi = (\varphi^1, \dots, \varphi^{N_0})$, where $\varphi^j = (\varphi^j_1, \dots, \varphi^j_{2K_0}) \in \mathbb{C}^{2K_0}$, $1 \leq j \leq N_0$, such that $\text{rank}(\Phi_\xi(\xi_0)) = \text{rank}(\varphi)$. In the following we suppress the evaluation of Φ at ξ_0 notationally and perform elementary row operations. The matrix given by

$$\begin{aligned} (\varphi^1, \dots, \varphi^{11}) &:= \left(\Phi_u, \Phi_{\bar{a}'_2}, \Phi_{c'_1}, \Phi_{c'_2}, \Phi_\lambda, \Phi_{\bar{c}}, \Phi_{a'_1}, \Phi_{r'}, \Phi_c, \Phi_{a'_2}, \Phi_s \right) \\ &\quad - \left(0, 0, 0, \Phi_u, \Phi_u, 0, \Phi_u, 0, \Phi_{c'_1}, i\varepsilon/2 \Phi_{\bar{c}}, 0 \right), \end{aligned}$$

is in row echelon form, with constant nonzero entries in the main diagonal. Each 0 above represents $0 \in \mathbb{C}^{2K_0}$. Next we define

$$\varphi^{12} := \Phi_{\lambda'} + \Phi_u/3 - \Phi_\lambda - \Phi_{a'_1}/3 - i\varepsilon/8 \Phi_{r'} + 10s_0/3 \Phi_s, \varphi^{13} := \Phi_{u'} - \Phi_u/3 - 2/3 \Phi_{a'_1} - 2/3 \Phi_s,$$

which are of the following form, where we denote by h' derivatives of a function h depending on s with respect to s :

$$\begin{aligned}\varphi^{12} &= (0, \dots, 0, \varphi_{12}^{12}, \dots, \varphi_{2K_0}^{12}) \\ &= \left(0, \dots, 0, \frac{-2(4x - 5s_0x')}{3}, 2i\varepsilon, \frac{8is_0}{3}, \frac{2i(3\varepsilon - 3y + 5s_0y')}{3}, -\frac{1}{3}, \varphi_{17}^{12}, \dots, \varphi_{2K_0}^{12}\right) \\ \varphi^{13} &= (0, \dots, 0, \varphi_{12}^{13}, \dots, \varphi_{2K_0}^{13}) = \left(0, \dots, 0, \frac{2x - s_0x'}{3}, 0, -\frac{8is_0}{3}, -\frac{is_0y'}{3}, -\frac{2}{3}, \varphi_{17}^{13}, \dots, \varphi_{2K_0}^{13}\right).\end{aligned}$$

Then we define $\varphi^{14} := \Phi_r - \Phi_{r'}$, $\varphi^{15} := \Phi_{\tilde{e}_2}$, $\varphi^{16} := \Phi_{\tilde{e}_1}$, and compute $\varphi^{14} = -2(e_{15} + e_{2K_0})$, $\varphi^{15} = e_{19}$, $\varphi^{16} = -2e_{24} + i\varepsilon e_{26} - 12\varepsilon se_{2K_0}$, where for $j \in \mathbb{N}$ we denote by e_j the j -th unit vector in \mathbb{R}^{2K_0} . We have to consider several cases. If $\varphi_{12}^{12} \neq 0$, we consider $\tilde{\varphi}^{13} := \varphi^{13} - \varphi_{12}^{13}\varphi^{12}/\varphi_{12}^{12}$, such that $\tilde{\varphi}_{13}^{13}$ is a multiple of $-2x + s_0x'$. If $\tilde{\varphi}_{13}^{13} \neq 0$, then $\varphi = (\varphi^1, \dots, \varphi^{12}, \tilde{\varphi}^{13}, \varphi^{14}, \varphi^{15}, \varphi^{16})$ is in echelon form. If $\tilde{\varphi}_{13}^{13} = 0$, then $x = Cs^2$, where $C \in \mathbb{C} \setminus \{0\}$ and we have $\tilde{\varphi}_{14}^{13} \neq 0$, which again implies that $\varphi = (\varphi^1, \dots, \varphi^{12}, \tilde{\varphi}^{13}, \varphi^{14}, \varphi^{15}, \varphi^{16})$ is in echelon form. Next we treat $\varphi_{12}^{12} = 0$. First we consider the trivial case. If $x = 0$, then since $s_0 > 0$, we have $x' = 0$ and $\varphi = (\varphi^1, \dots, \varphi^{16})$ is in echelon form. Now we assume $x \neq 0$ which implies $x' \neq 0$ and we solve $\varphi_{12}^{12} = 0$. The solution is given by $x = Cs^{4/5}$, where $C \in \mathbb{C} \setminus \{0\}$ and $\varphi = (\varphi^1, \dots, \varphi^{11}, \varphi^{13}, \varphi^{12}, \varphi^{14}, \varphi^{15}, \varphi^{16})$ is in echelon form. We sum up that in all cases the Jacobian $\Phi_\xi(\xi_0)$ of the system Φ evaluated at ξ_0 is of full rank N_0 , hence we conclude that Ψ from (7.1) is a real-analytic locally regular mapping if we choose $\delta_0 > 0$ sufficiently small in M . For Ψ to be a local parametrization of M it remains to show that for each sufficiently small neighborhood $U \subset \Xi \subset \mathbb{R}^{N_0}$ of ξ_0 , there exists a neighborhood $W \subset \mathbb{C}^{K_0}$ of $\Psi(\xi_0) = F_{s_0}$, such that $\Psi(U) = W \cap M$. We have $\Psi(U) = \{j_0(H) : \exists \xi = (\gamma, \gamma', t) \in U : H = \phi_{\gamma'}^{-1} \circ F_t \circ \phi_\gamma\}$ and with the notation from the very beginning of this proof for $\delta > 0$ we have

$$M = \Pi^{-1}(F_{\delta, s_0}) = \{H \in \mathfrak{F} : \exists (\gamma, \gamma', s) \in \Gamma \times \Gamma' \times B_\delta(s_0) \cap \mathbb{R}^+ : \phi_{\gamma'}^{-1} \circ H \circ \phi_\gamma^{-1} = F_s\}.$$

Remark 2.15, together with the fact that for each $H \in M$ we can write $H = \phi_{\gamma'}^{-1} \circ F_s \circ \phi_\gamma$, shows $\Psi(U) \subset M$. We assume that there exists $U \subset \Xi$ a neighborhood of ξ_0 , such that for any neighborhood W of $\Psi(\xi_0) = F_{s_0}$ we have $\Psi(U) \neq W \cap M$. We choose open, connected neighborhoods $(W_n)_{n \in \mathbb{N}}$ of F_{s_0} with $\bigcap_n W_n = \{F_{s_0}\}$ and $\Psi(U) \neq W_n \cap M$ for all $n \in \mathbb{N}$. There exists a sequence of mappings $(H_n)_{n \in \mathbb{N}} \in \mathfrak{F}$ such that $H_n \in W_n \cap M$ and $H_n \notin \Psi(U)$. We write $H_n = \phi_{\gamma_n'}^{-1} \circ F_{s_n} \circ \phi_{\gamma_n}$ and conclude by Lemma 6.3 that $(\gamma_n, \gamma_n', s_n) \rightarrow \xi_0$ in Ξ . Thus eventually $H_n \in \Psi(U)$ for large enough $n \in \mathbb{N}$, which completes the proof of the lemma. \square

Remark 7.2. By [BER97, Corollary 1.2] the group \mathcal{G} is a totally real, closed, real-analytic submanifold of $G_0^2(\mathbb{H}^2, 0) \times G_0^2(\mathbb{H}_\varepsilon^3, 0) \subset J_0^2(\mathbb{H}^2, 0) \times J_0^2(\mathbb{H}_\varepsilon^3, 0)$. Hence \mathcal{G} is a real-analytic real Lie group. [BM45, Theorem 4] states that the action of a real-analytic Lie group G on a real-analytic manifold M is real-analytic, i.e., the map $G \times M \rightarrow M, (g, m) \rightarrow g \cdot m$ is a real-analytic map between real-analytic manifolds. Hence we obtain for M being a real-analytic submanifold, that $N : \mathcal{G} \times M \rightarrow M$ is a real-analytic action.

Theorem 7.3 ([DK00, Theorem 1.11.4]). *Let M be a real-analytic manifold equipped with an action $G \times M \rightarrow M$, where G is a real-analytic Lie group. Assume that the action is free and proper. Then M/G has the unique structure of a real-analytic manifold of real dimension $\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} G$ and the topology of M/G is the quotient topology τ_Q . We denote by $\varphi : M \rightarrow M/G$ the canonical projection given by $\varphi(m) = G \cdot m := \{g \cdot m : g \in G\}$ for $m \in M$. For every $s \in M/G$ there is an open neighborhood $S \subset M/G$ of s and a real-analytic diffeomorphism $\psi : \varphi^{-1}(S) \rightarrow G \times S, \psi : m \mapsto (\psi_1(m), \psi_2(m))$, such that for $m \in \varphi^{-1}(S), g \in G$ we have $\varphi(m) = \psi_2(m)$ and $\psi(g \cdot m) = (g \cdot \psi_1(m), \psi_2(m))$.*

Remark 7.4. The above [Theorem 7.3](#) says that the triple $(\varphi, M, M/G)$ is a *real-analytic principle fibre bundle with structure group G* .

Proof of Theorem 1.4. We note that by [Lemma 4.1](#) and [Lemma 7.1](#) the set \mathfrak{F} is a real-analytic manifold and by [Remark 7.2](#) we know that \mathcal{G} is a real-analytic Lie group. Thus from [Theorem 7.3](#) and [Remark 7.4](#) the conclusion for $\varepsilon = +1$ follows. Next we show the claim for $\varepsilon = -1$: For $k=1,2$ we set $N_k := \{G_{k+1,s}^- : s > 0\}$ and $N_0 := N_1 \cap N_2 = \{G_{2,1/2}^-\}$. The corresponding preimages are denoted by $M_k := \Pi^{-1}(N_k) \subset \mathfrak{F}$, such that $M_0 := M_1 \cap M_2 = \Pi^{-1}(N_0)$. We set $M := M_1 \cup M_2$. By [Lemma 7.1](#) for $k = 1, 2$ we have that M_k is a real-analytic submanifold of \mathfrak{F} . Thus by [Theorem 7.3](#) locally M_k is real-analytically diffeomorphic to $\mathcal{G} \times S_k$, where S_k is a real submanifold with $\dim_{\mathbb{R}}(S_k) = \dim_{\mathbb{R}}(M_k) - \dim_{\mathbb{R}}(\mathcal{G}) = 1$, by [Lemma 7.1](#). By [Proposition 2.6](#) it is possible to normalize any element in S_k with unique isotropies which depend real-analytically on elements of S_k . Thus, since $\dim_{\mathbb{R}}(N_k) = 1$, we map S_k to N_k via real-analytic diffeomorphisms. We obtain that for $k = 1, 2$ there exists an open neighborhood $U_k \subset \mathfrak{F}$ of N_0 and a real-analytic diffeomorphism $\phi_k : U_k \rightarrow V_k$ such that $\phi_k(U_k \cap M_k) = (\mathcal{G} \times N_k) \cap V_k$, where V_k is an open neighborhood of $N'_0 := \{\text{id}\} \times N_0 \subset \mathcal{G} \times M$, where $\text{id} = (\text{id}_{\mathbb{C}^2}, \text{id}_{\mathbb{C}^3})$. Moreover $\phi_k(U_k \cap N_k) = (\{\text{id}\} \times N_k) \cap V_k$ and ϕ_k satisfies the properties given in [Theorem 7.3](#). We define $\phi : U_0 \rightarrow V_0$, $\phi(x) := \phi_k(x)$ for $x \in U_0 \cap U_k$, where $k = 1, 2$ and $V_0 = V_1 \cup V_2$ is an open neighborhood of N'_0 . Write $\tilde{U} := U_1 \cap U_2 \cap U_0 \subset \mathfrak{F}$ for an open neighborhood of N_0 . Then we have $\phi|_{\tilde{U}} = \phi_1|_{\tilde{U}} = \phi_2|_{\tilde{U}}$, which implies that ϕ is a real-analytic diffeomorphism. Furthermore, since $\text{image}(\phi_1|_{\tilde{U} \cap M}) = \text{image}(\phi_2|_{\tilde{U} \cap M}) = (\mathcal{G} \times N_0) \cap \tilde{V}$, where \tilde{V} is an open neighborhood of $N'_0 \subset \mathcal{G} \times M$, the mapping ϕ locally maps M_0 real-analytically diffeomorphic to $\mathcal{G} \times N_0$. Finally the last statement follows from [Theorem 7.3](#), since if \mathfrak{F} would be a smooth manifold, then the quotient \mathfrak{N} needs to be a smooth manifold, by the smooth version of [Theorem 7.3](#), see also [\[DK00, Theorem 1.11.4\]](#), which is not the case. \square

Proof of Theorem 1.5. We show that $\Pi : \mathfrak{F} \rightarrow \mathfrak{N}$ is a surjective, continuous and closed mapping with respect to τ_J . Surjectivity is clear from [Proposition 2.6](#) and [Theorem 2.8](#). To show continuity of Π with respect to τ_J we let $(H_n)_{n \in \mathbb{N}}$ be a sequence of mappings in \mathfrak{F} and $H \in \mathfrak{F}$, such that $H_n \rightarrow H$. Assuming w.l.o.g. that $H \in \mathfrak{N}$ we need to conclude that $\Pi(H_n) \rightarrow H$. We have $\Pi(H_n) = \phi'_n \circ H_n \circ \phi_n^{-1} \in \mathfrak{N}$, where $(\phi_n, \phi'_n) \in \mathcal{G}$ are the isotropies according to [Proposition 2.6](#). Assume $\phi'_n \circ H_n \circ \phi_n^{-1} \rightarrow \hat{H} \in \mathfrak{N}$, then by [Lemma 6.3](#) we obtain $(\phi_n, \phi'_n) \rightarrow (\text{id}_{\mathbb{C}^2}, \text{id}_{\mathbb{C}^3})$ and since $H_n \rightarrow H$ we get $\hat{H} = H$. We are left by proving the closedness of Π with respect to τ_J : Let $C \subset \mathfrak{F}$ be a closed subset. We need to show that $\Pi(C) \subset \mathfrak{N}$ is a closed subset. To prove this statement we let $H_n \in \Pi(C)$ for $n \in \mathbb{N}$, forming a sequence of mappings in \mathfrak{N} such that $H_n \rightarrow H_0$, where $H_0 \in \mathfrak{N}$. For the closedness of $\Pi(C)$ we need to conclude that $H_0 \in \Pi(C)$. By [Theorem 2.8](#) we can write $H_n = G_{k_n, s_n}^\varepsilon$ and $H_0 = G_{k_0, s_0}^\varepsilon$ for $k_n, k_0 \in \{2, 3\}$. Note that since $H_n \rightarrow H_0$ in \mathfrak{N} we have $s_n \rightarrow s_0$. This implies that for any convergent sequence $G_n \in \Pi^{-1}(H_n)$ the map $G_0 := \lim_{n \rightarrow \infty} G_n$ belongs to $\Pi^{-1}(H_0)$. Since C is closed, an arbitrary convergent sequence $F_n \in \Pi^{-1}(H_n) \cap C$ with $F_n \rightarrow F_0$ thus satisfies $F_0 \in \Pi^{-1}(H_0) \cap C$, which implies $H_0 = \Pi(F_0) \in \Pi(C)$. \square

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Texas A&M University at Qatar, PO Box 23874, Doha, Qatar
 michael.reiter@qatar.tamu.edu